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Eigenvalues of Casimir operators for $gl(m/\infty)$

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Abstract. A full set of Casimir operators for the Lie superalgebra $gl(m/\infty)$ is constructed and shown to be well defined in the category O_{FS} generated by the highest-weight irreducible representations with only a finite number of non-zero weight components. The eigenvalues of these Casimir operators are determined explicitly in terms of the highest weight. Characteristic identities satisfied by certain (infinite) matrices with entries from $gl(m/\infty)$ are also determined.

1. Introduction

During the last few years the infinite-dimensional Lie algebras and Lie superalgebras have played an important role in several areas of theoretical and mathematical physics [1–9]. They have applications in the theory of integrable field equations, string theory, two-dimensional statistical models. In addition, these algebras are of interest as examples of Kac–Moody Lie (super-)algebras of infinite type.

However, for these algebras such a fundamental concept as Casimir invariants has not yet been determined. The present paper is a step toward solving this problem.

We construct a full set of Casimir operators for the infinite-dimensional general linear Lie superalgebra $gl(m/\infty)$ corresponding to the natural matrix realization, namely

$$gl(m/\infty) = \left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \, A \in M_{m\times m}, B \in M_{m\times\infty}, C \in M_{\infty\times m}, D \in M_{\infty\times\infty}, \right. \\ \left. \text{all but a finite number of } X_{ij} \in \mathbb{C} \text{ are zero} \right\} \tag{1}$$

where $M_{p\times q}$ is the space of all $p \times q$ complex matrices. The even subalgebra $gl(m/\infty)_{\bar{0}}$ has $B = 0$ and $C = 0$; the odd subspace $gl(m/\infty)_{\bar{1}}$ has $A = 0$ and $D = 0$.

A basis for the Lie superalgebra $gl(m/\infty)$ is given by the Weyl generators E_{ij}, $i, j = -m+1, -m+2, \ldots, 0, 1, \ldots$. Assign to each index i a degree $\langle i \rangle$, which is zero for $i \in -\mathbb{Z}_+$ and 1 for $i \in \mathbb{N}$ (see the notation at the end of the introduction). Then the generator E_{ij} is even (respectively odd), if $\langle i \rangle + \langle j \rangle$ is an even (respectively odd) number. The multiplication ($\equiv$ the supercommutator) $[\![\, ,]\!]$ of $gl(m/\infty)$ is given by the linear extension of the relations

$$[\![E_{ij}, E_{kl}]\!] = \delta_{jk} E_{il} - (-1)^{(\langle i \rangle + \langle j \rangle)(\langle k \rangle + \langle l \rangle)} \delta_{il} E_{kj}. \tag{2}$$

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We will consider the category O_{FS} generated by all highest-weight irreducible $gl(m/\infty)$ modules $V(\Lambda)$ with a finite number of non-zero highest-weight components Λ_i of the highest weight

$$\begin{aligned}\Lambda &\equiv (\Lambda_{-m+1}, \Lambda_{-m+2}, \dots, \Lambda_0, \Lambda_1, \dots, \Lambda_k, 0, 0, \dots) \\ &\equiv (\Lambda_{-m+1}, \Lambda_{-m+2}, \dots, \Lambda_0, \Lambda_1, \dots, \Lambda_k, \dot{0}).\end{aligned} \tag{3}$$

The highest-weight Λ of $V(\Lambda)$ uniquely characterized the module and satisfies the conditions:

$$\Lambda_i - \Lambda_{i+1} \in \mathbb{Z}_+ \qquad \forall i \neq 0. \tag{4}$$

Denote by H the Cartan subalgebra of $gl(m/\infty)$. The dual space H^* of H is described by the forms ε_i, $i = -m+1, -m+2, \dots$, where ε_i: $X \to A_{ii}$, for $-m+1 \leqslant i \leqslant 0$ and ε_i: $X \to D_{ii}$, $\forall i \in \mathbb{N}$, and X is given by (1) only for diagonal X. On H^* there is a bilinear form (,) defined by

$$\begin{aligned}&(\epsilon_i, \epsilon_j) = \delta_{ij} && \text{for} \quad -m+1 \leqslant i, \quad j \leqslant 0 \\ &(\epsilon_i, \epsilon_j) = 0 && \text{for} \quad -m+1 \leqslant i \leqslant 0 \quad \text{and} \quad j \in \mathbb{N} \\ &(\epsilon_i, \epsilon_j) = -\delta_{ij} && \text{for} \quad i, j \in \mathbb{N}.\end{aligned} \tag{5}$$

The roots $\varepsilon_i - \varepsilon_j$ $(i \neq j)$ of $gl(m/\infty)$ are the non-zero weights of the adjoint representation. The positive roots are those given by the set

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j | \; i < j, \; i, j = -m+1, -m+2, \dots\}. \tag{6}$$

Define

$$\rho = \tfrac{1}{2} \sum_{i=-m+1}^{0} (1 - 2i - 2m)\epsilon_i + \tfrac{1}{2} \sum_{i=1}^{\infty} (1 - 2i + 2m)\epsilon_i. \tag{7}$$

Let D_n be the set of $gl(m/\infty)$ weights

$$\begin{aligned}D_n = \{\nu | \; &\nu = (\nu_{-m+1}, \dots, \nu_0, \nu_1, \dots, \nu_n, \dot{0}), \\ &\nu_i \in \mathbb{Z}_+, \; i = -m+1, -m+2, \dots, n-1, \; \nu_n \in \mathbb{N}\}\end{aligned} \tag{8}$$

and let $D_n^+ \subset D_n$ be the subset of integral dominant weights in D_n:

$$D_n^+ = \{\nu | \; \nu \in D_n, \; \nu_i - \nu_{i+1} \in \mathbb{Z}_+, \; \forall i \neq 0\}. \tag{9}$$

Note that if ν is a weight in $V(\Lambda)$, $\Lambda \in D_k^+$, then $\nu \in D_n$, for some $n \in \mathbb{Z}_+$.

In section 2 we construct a full set of Casimir operators convergent on each module $V(\Lambda)$. The eigenvalues of these Casimir invariants for all modules from the category O_{FS} are computed in section 3. In section 4 we present a derivation of the polynomial identities satisfied by certain matrices with entries from $gl(m/\infty)$.

Throughout the paper we use the following notation:

- irrep(s), irreducible representation(s);
- $\mathbb{C}$, the complex numbers;
- $\mathbb{Z}_+$, all non-negative integers;
- $\mathbb{N}$, all positive integers;
- $U(A)$, the universal enveloping algebra of A;
- $$\langle i \rangle = \begin{cases} 0 & \text{for} \quad i \in -\mathbb{Z}_+ \\ 1 & \text{for} \quad i \in \mathbb{N}. \end{cases}$$

2. Construction of Casimir operators

An obvious invariant for $gl(m/\infty)$ is the first-order invariant

$$I_1 = \sum_{i=-m+1}^{\infty} E_{ii}. \tag{10}$$

It is not clear, however, how to construct appropriate higher-order Casimir operators for $gl(m/\infty)$. Let us first consider the second-order invariant $I_2^{(m,n)}$ of $gl(m/n)$:

$$\begin{aligned}
I_2^{(m,n)} &= \sum_{i,j=-m+1}^{n} (-1)^{\langle j\rangle} E_{ij}E_{ji} \\
&= \sum_{i,j=-m+1}^{0} E_{ij}E_{ji} - \sum_{i,j=1}^{n} E_{ij}E_{ji} + \sum_{i=1}^{n}\sum_{j=-m+1}^{0} E_{ij}E_{ji} - \sum_{i=-m+1}^{0}\sum_{j=1}^{n} E_{ij}E_{ji} \\
&= \sum_{i=-m+1}^{0}\sum_{j<i=-m+1}^{0} E_{ij}E_{ji} + \sum_{i=-m+1}^{0}\sum_{j>i=-m+1}^{0} E_{ij}E_{ji} + \sum_{i=-m+1}^{0} E_{ii}^2 \\
&\quad - \sum_{i=1}^{n}\sum_{j<i=1}^{n} E_{ij}E_{ji} - \sum_{i=1}^{n}\sum_{j>i=1}^{n} E_{ij}E_{ji} - \sum_{i=1}^{n} E_{ii}^2 + 2\sum_{i=1}^{n}\sum_{j=-m+1}^{0} E_{ij}E_{ji} \\
&\quad - \sum_{i=-m+1}^{0}\sum_{j=1}^{n}(E_{ii}+E_{jj}) \\
&= 2\sum_{i=-m+1}^{0}\sum_{j<i=-m+1}^{0} E_{ij}E_{ji} + \sum_{i=-m+1}^{0}\sum_{j>i=-m+1}^{0}(E_{ii}-E_{jj}) + \sum_{i=-m+1}^{0} E_{ii}^2 \\
&\quad - 2\sum_{i=1}^{n}\sum_{j<i=1}^{n} E_{ij}E_{ji} - \sum_{i=1}^{n}\sum_{j>i=1}^{n}(E_{ii}-E_{jj}) - \sum_{i=1}^{n} E_{ii}^2 + 2\sum_{i=1}^{n}\sum_{j=-m+1}^{0} E_{ij}E_{ji} \\
&\quad - n\sum_{i=-m+1}^{0} E_{ii} - m\sum_{i=1}^{n} E_{ii} \\
&= 2\sum_{i=-m+1}^{n}\sum_{j<i=-m+1}^{n} (-1)^{\langle j\rangle} E_{ij}E_{ji} + \sum_{i=-m+1}^{0} E_{ii}(E_{ii}+1-m-2i) \\
&\quad - \sum_{i=1}^{n} E_{ii}(E_{ii}+1+n-2i) - n\sum_{i=-m+1}^{0} E_{ii} - m\sum_{i=1}^{n} E_{ii} \\
&= 2\sum_{i=-m+1}^{n}\sum_{j<i=-m+1}^{n} (-1)^{\langle j\rangle} E_{ij}E_{ji} + \sum_{i=-m+1}^{n} (-1)^{\langle i\rangle} E_{ii}(E_{ii}+1-2i) - (m+n)I_1^{(m,n)} \\
&= 2\sum_{i=-m+1}^{n}\sum_{j<i=-m+1}^{n} (-1)^{\langle j\rangle} E_{ij}E_{ji} \\
&\quad + \sum_{i=-m+1}^{n} (-1)^{\langle i\rangle} E_{ii}(E_{ii}+1-2i) - 2mI_1^{(m,n)} + (m-n)I_1^{(m,n)}
\end{aligned} \tag{11}$$

where $I_1^{(m,n)} \equiv \sum_{i=-m+1}^{n} E_{ii}$ is the first-order invariant of $gl(m/n)$. Due to the last term in (11) the $gl(m/n)$ second-order invariant diverges as $n \to \infty$. Eliminating the last term in (11) (the rest of the expression is also an invariant) and taking the limit $n \to \infty$ one obtains the following quadratic Casimir for $gl(m/\infty)$:

$$I_2 = 2\sum_{i=-m+1}^{\infty}\sum_{j<i=-m+1}^{\infty}(-1)^{\langle j\rangle}E_{ij}E_{ji} + \sum_{i=-m+1}^{\infty}(-1)^{\langle i\rangle}E_{ii}(E_{ii}+1-2i) - 2mI_1 \tag{12}$$

which is convergent (see formula (21)) on the category O_{FS} of irreps considered. On $V(\Lambda)$, $\Lambda \in D_k^+$, I_2 takes constant value

$$\chi_\Lambda(I_2) = \sum_{i=-m+1}^{k}\left((-1)^{\langle i\rangle}\Lambda_i(\Lambda_i+1-2i) - 2m\Lambda_i\right) = (\Lambda, \Lambda+2\rho). \tag{13}$$

This consideration shows how to construct the higher-order Casimir operators of $gl(m/\infty)$.

Introduce to this end the characteristic matrix

$$A_i{}^j = (-1)^{\langle i\rangle\langle j\rangle}E_{ji}. \tag{14}$$

Define the powers of the matrix A recursively by

$$\left(A^q\right)_i{}^j = \sum_{k=-m+1}^{\infty}A_i{}^k\left(A^{q-1}\right)_k{}^j \qquad \left(\left(A^0\right)_i{}^j \equiv \delta_{ij}\right). \tag{15}$$

Using induction and the $gl(m/\infty)$ commutation relations (2) one obtains:

Proposition 1.

$$[\![E_{kl}, \left(A^q\right)_i{}^j]\!] = (-1)^{(\langle k\rangle+\langle l\rangle)\langle i\rangle}\left(\delta_{lj}\left(A^q\right)_i{}^k - \delta_{ik}\left(A^q\right)_l{}^j\right). \tag{16}$$

Therefore the matrix supertraces

$$str\left(A^q\right) \equiv \sum_{i=-m+1}^{\infty}(-1)^{\langle i\rangle}\left(A^q\right)_i{}^i \tag{17}$$

are formally Casimir operators. They are, however, divergent except for $q = 1$ in which case we obtain the first-order invariant (10). Our purpose is to construct a full set of Casimir invariants which are well defined and convergent on the category O_{FS}.

Theorem 1. *The Casimir operators defined recursively by*

$$\begin{aligned} I_1 &= \sum_{i=-m+1}^{\infty}(-1)^{\langle i\rangle}A_i{}^i = str(A) \\ I_q &= \sum_{i=-m+1}^{\infty}(-1)^{\langle i\rangle}\left[\left(A^q\right)_i{}^i - I_{q-1}\right] = str\left[A^q - I_{q-1}\right] \qquad q = 2, 3, \ldots \end{aligned} \tag{18}$$

form a full set of convergent $gl(m/\infty)$ Casimir operators on each module $V(\Lambda) \in O_{FS}$.

Observe that the operators I_q are indeed Casimir invariants (see proposition 1). Then it remains to prove they are convergent on the category O_{FS}. We will do this by induction. Consider first the case $q = 2$:

$$\begin{aligned} I_2 &\equiv \sum_{j=-m+1}^{\infty}(-1)^{\langle j\rangle}\left[\left(A^2\right)_j{}^j - I_1\right] = \sum_{j=-m+1}^{0}\left[\sum_{i=-m+1}^{\infty}E_{ij}E_{ji} - I_1\right] \\ &\quad - \sum_{j=1}^{\infty}\left[\sum_{i=-m+1}^{\infty}E_{ij}E_{ji} - I_1\right] = \sum_{j=-m+1}^{0}\sum_{i=-m+1}^{0}E_{ij}E_{ji} + \sum_{j=-m+1}^{0}\sum_{i=1}^{\infty}E_{ij}E_{ji} - mI_1 \\ &\quad - \sum_{j=1}^{\infty}\sum_{i=-m+1}^{0}E_{ij}E_{ji} - \sum_{j=1}^{\infty}\left[\sum_{i=1}^{\infty}E_{ij}E_{ji} - I_1\right] \end{aligned}$$

$$= 2\sum_{i=-m+1}^{0}\sum_{j<i=-m+1}^{0} E_{ij}E_{ji} + \sum_{i=-m+1}^{0} E_{ii}(E_{ii}+1-m-2i) - mI_1 + 2\sum_{j=-m+1}^{0}\sum_{i=1}^{\infty} E_{ij}E_{ji}$$

$$- \sum_{j=1}^{\infty}\sum_{i=-m+1}^{0}(E_{ii}+E_{jj}) - \sum_{j=1}^{\infty}\left[2\sum_{i>j=1}^{\infty}E_{ij}E_{ji} + \sum_{i<j=1}^{\infty}(E_{ii}-E_{jj}) + E_{jj}^2 - I_1\right]$$

$$= 2\sum_{i=-m+1}^{\infty}\sum_{j<i=-m+1}^{\infty}(-1)^{\langle j\rangle}E_{ij}E_{ji} + \sum_{i=-m+1}^{0}E_{ii}(E_{ii}+1-m-2i) - mI_1 - m\sum_{i=1}^{\infty}E_{ii}$$

$$- \sum_{j=1}^{\infty}\left[\sum_{i<j=1}^{\infty}(E_{ii}-E_{jj}) + E_{jj}^2 - \sum_{i=1}^{\infty}E_{ii}\right]$$

$$= 2\sum_{i=-m+1}^{\infty}\sum_{j<i=-m+1}^{\infty}(-1)^{\langle j\rangle}E_{ij}E_{ji} + \sum_{i=-m+1}^{0}E_{ii}(E_{ii}+1-2i) - 2mI_1$$

$$- \sum_{i=1}^{\infty}E_{ii}(E_{ii}+1-2i) \tag{19}$$

which agrees with the definition (12).

Now let $v \in V(\Lambda)$, $\Lambda \in D_k^+$, be an arbitrary weight vector. Then the weight of v has the form

$$\nu = (\nu_{-m+1}, \nu_{-m+2}, \ldots, \nu_0, \ldots, \nu_r, \dot{0}). \tag{20}$$

Since

$$A_i{}^j v = (-1)^{\langle i\rangle\langle j\rangle}E_{ji}v = 0 \qquad \forall i > r \tag{21}$$

the second-order invariant I_2 is convergent on each $V(\Lambda) \in O_{FS}$ (cf formula (13)).

Applying proposition 1 and (21), for $i > r$ one obtains

$$\begin{aligned}\left(A^q\right)_i{}^i v &= \sum_{j=-m+1}^{\infty} A_i{}^j\left(A^{q-1}\right)_j{}^i v = \sum_{j=-m+1}^{\infty}(-1)^{\langle i\rangle\langle j\rangle}E_{ji}\left(A^{q-1}\right)_j{}^i v\\ &= \sum_{j=-m+1}^{\infty}(-1)^{\langle i\rangle\langle j\rangle}\left\{(-1)^{(\langle j\rangle+\langle i\rangle)\langle j\rangle}\left[\left(A^{q-1}\right)_j{}^j - \left(A^{q-1}\right)_i{}^i\right]v\right.\\ &\quad \left.+ (-1)^{(\langle i\rangle+\langle j\rangle)}\left(A^{q-1}\right)_j{}^i E_{ji}v\right\}\\ &= \sum_{j=-m+1}^{\infty}(-1)^{\langle j\rangle}\left[\left(A^{q-1}\right)_j{}^j - \left(A^{q-1}\right)_i{}^i\right]v.\end{aligned} \tag{22}$$

For the case $q = 2$ we have

$$\left(A^2\right)_i{}^i v = \sum_{j=-m+1}^{\infty}(-1)^{\langle j\rangle}\left[A_j{}^j - A_i{}^i\right]v = \sum_{j=-m+1}^{\infty}E_{jj}v = I_1 v \qquad \forall i > r \tag{23}$$

so that

$$\left(\left(A^2\right)_i{}^i - I_1\right)v = 0 \qquad \forall i > r \tag{24}$$

which is another proof for the convergence of I_2. More generally

Proposition 2. *For any weight vector $v \in V(\Lambda)$, and $q = 2, 3, \ldots$ there exist $r \in \mathbb{N}$ such that*

$$\left(\left(A^q\right)_i{}^i - I_{q-1}\right)v = 0 \qquad \forall i > r. \tag{25}$$

Proof. We proceed by induction. Assume v has weight ν as in (20). Formula (25) is valid for $q = 2$ (24). Let the result be true for a given q, i.e.

$$\left(A^q\right)_i{}^i v = I_{q-1} v \qquad \forall i > r.$$

Then (see equation (22))

$$\begin{aligned}\left(A^{q+1}\right)_i{}^i v &= \sum_{j=-m+1}^{\infty} (-1)^{\langle j\rangle}\left[\left(A^q\right)_j{}^j - \left(A^q\right)_i{}^i\right] v \\ &= \sum_{j=-m+1}^{\infty} (-1)^{\langle j\rangle}\left[\left(A^q\right)_j{}^j - I_{q-1}\right] v = I_q v \qquad \forall i > r \end{aligned} \tag{26}$$

which proves (25). □

I_q (18) is convergent on each $V(\Lambda)$ for $q = 2$. Assume it is well defined and convergent on $V(\Lambda)$ for a given q. Then, with v as in (25), we have

$$\begin{aligned} I_{q+1} v &\equiv \sum_{i=-m+1}^{\infty} (-1)^{\langle i\rangle}\left[\left(A^{q+1}\right)_i{}^i - I_q\right] v = \sum_{i=-m+1}^{r} (-1)^{\langle i\rangle}\left[\left(A^{q+1}\right)_i{}^i - I_q\right] v \\ &= \sum_{i=-m+1}^{r} (-1)^{\langle i\rangle}\left(A^{q+1}\right)_i{}^i v + (r-m) I_q v. \end{aligned} \tag{27}$$

Therefore I_{q+1} is convergent and well defined on $V(\Lambda)$.

This completes the (inductive) proof of theorem 1.

3. Eigenvalue formula for Casimir operators

In this section we apply our previous results to evaluate the spectrum of the operators (18).

Let $v \in V(\Lambda)$, be an arbitrary vector of weight $\nu = (\nu_{-m+1}, \nu_{-m+2} \ldots, \nu_0, \nu_1, \ldots, \nu_r, \dot{0})$. Then, keeping in mind proposition 1, the fact that $\left(A^{q-1}\right)_k{}^j$ has weight $\varepsilon_j - \varepsilon_k$ under the adjoint representation of $gl(m/\infty)$ and that all vectors of $V(\Lambda)$ have weight components ν_i in $\mathbb{Z}_+$, we must have for $j \leqslant r$

$$\left(A^{q-1}\right)_k{}^j v = 0 \qquad \forall k > r. \tag{28}$$

Therefore

$$\left(A^q\right)_i{}^j v = \sum_{k=-m+1}^{\infty} A_i{}^k \left(A^{q-1}\right)_k{}^j v = \sum_{k=-m+1}^{r} A_i{}^k \left(A^{q-1}\right)_k{}^j v. \tag{29}$$

Proceeding recursively we may therefore write

$$\left(A^q\right)_i{}^j v = \left(\bar{A}^q\right)_i{}^j v \qquad \forall i, j = -m+1, -m+2, \ldots, r \tag{30}$$

where $(\bar{A})_i{}^j = (-1)^{\langle i\rangle\langle j\rangle} E_{ji}, \forall i, j = -m+1, \ldots, r,$ is the $gl(m/r)$ characteristic matrix, and the powers of the matrix $\bar{A}$ are defined by (15) with $i, j, k = -m+1, \ldots, r$ and $\bar{A}$ instead of A. It follows then that the formula (27) can be written as

$$I_q v = \sum_{i=-m+1}^{r} (-1)^{\langle i\rangle}\left[\left(\bar{A}^q\right)_i{}^i - I_{q-1}\right] v = \left[I_q{}^{(m,r)} - (m-r) I_{q-1}\right] v \tag{31}$$

with

$$I_q{}^{(m,r)} = \sum_{i=-m+1}^{r} (-1)^{\langle i\rangle}\left(\bar{A}^q\right)_i{}^i \tag{32}$$

being the qth-order invariant of $gl(m/r)$. formula (31) is valid $\forall q \in \mathbb{N}$, which gives a recursion relation for the I_q with initial condition

$$I_1 v = \chi_\Lambda(I_1) v. \tag{33}$$

In particular, it follows from (31) that the invariants I_q are certainly convergent on all weight vectors $v \in V(\Lambda)$.

To determine the eigenvalues of I_q let $v = v_\Lambda^+$ be the highest-weight vector of the $V(\Lambda)$ module and let

$$\Lambda = (\bar{\Lambda}, \dot{0}) \in D_k^+ \qquad \bar{\Lambda} \equiv (\Lambda_{-m+1}, \Lambda_{-m+2}, \ldots, \Lambda_0, \Lambda_1, \ldots, \Lambda_k). \tag{34}$$

Then for the eigenvalues of the I_q one obtains the recursion relation (see equation (31)):

$$\chi_\Lambda(I_q) = \chi_{\bar{\Lambda}}\big({I_q}^{(m,k)}\big) - (m-k)\chi_\Lambda(I_{q-1}) \qquad \chi_\Lambda(I_1) = \sum_{i=-m+1}^{k} \Lambda_i \tag{35}$$

where $\chi_{\bar{\Lambda}}\big({I_q}^{(m,k)}\big)$ is the eigenvalue of the qth-order invariant (32) of $gl(m/k)$ on the irreducible $gl(m/k)$ module with highest-weight $\bar{\Lambda}$; the latter is given explicitly by [10]

$$\chi_{\bar{\Lambda}}\big({I_q}^{(m,k)}\big) = \sum_{i=-m+1}^{k} (-1)^{\langle i\rangle} \alpha_i^q \prod_{j\neq i=-m+1}^{k} \left(\frac{\alpha_i - \alpha_j + (-1)^{\langle j\rangle}}{\alpha_i - \alpha_j} \right) \tag{36}$$

where

$$\alpha_i = (-1)^{\langle i\rangle}(\Lambda_i - i + 1) - m.$$

Therefore we obtain for the eigenvalues of the Casimir operators I_q

$$\chi_\Lambda(I_q) = \sum_{i=-m+1}^{k} (-1)^{\langle i\rangle} P_q(\alpha_i) \prod_{j\neq i=-m+1}^{k} \left(\frac{\alpha_i - \alpha_j + (-1)^{\langle j\rangle}}{\alpha_i - \alpha_j} \right) \tag{37}$$

for suitable polynomials $P_q(x)$ which, from equation (35), satisfy the recursion relation

$$P_q(x) = x^q - (m-k)P_{q-1}(x) \qquad P_1(x) = x. \tag{38}$$

In particular,

$$P_2(x) = x^2 - (m-k)x = x\frac{x^2 - (m-k)^2}{x + (m-k)} \tag{39a}$$

$$P_3(x) = x^3 - (m-k)\big(x^2 - (m-k)x\big) = x\frac{x^3 + (m-k)^3}{x + (m-k)} \tag{39b}$$

and more generally, it is easily established by induction that

$$P_q(x) = x\frac{x^q - (-1)^q(m-k)^q}{x + (m-k)}. \tag{40}$$

Thus we have

Theorem 2. *The eigenvalues of the Casimir operators I_q (18), on the irreducible $gl(m/\infty)$ module $V(\Lambda)$, $\Lambda \in D_k^+$ are given by*

$$\chi_\Lambda(I_q) = \sum_{i=-m+1}^{k} (-1)^{\langle i\rangle} \alpha_i \left(\frac{\alpha_i^q - (-1)^q(m-k)^q}{\alpha_i + (m-k)} \right) \prod_{j\neq i=-m+1}^{k} \left(\frac{\alpha_i - \alpha_j + (-1)^{\langle j\rangle}}{\alpha_i - \alpha_j} \right) \tag{41}$$

where $\alpha_i = (-1)^{\langle i\rangle}\,(\Lambda_i - i + 1) - m$.

4. Polynomial identities

Let Δ be the comultiplication on the enveloping algebra $U[gl(m/\infty)]$ of $gl(m/\infty)$ ($\Delta(E_{ij}) = E_{ij} \otimes 1 + 1 \otimes E_{ij}, i, j = -m+1, -m+2, \ldots$ with 1 being the unit in $U[gl(m/\infty)]$). Applying Δ to the second-order Casimir operator (12) of $gl(m/\infty)$ we obtain

$$\Delta(I_2) = I_2 \otimes 1 + 1 \otimes I_2 + 2 \sum_{i,j=-m+1}^{\infty} (-1)^{\langle j \rangle} E_{ij} \otimes E_{ji}. \tag{42}$$

Therefore

$$\sum_{i,j=-m+1}^{\infty} (-1)^{\langle j \rangle} E_{ij} \otimes E_{ji} = \tfrac{1}{2}[\Delta(I_2) - I_2 \otimes 1 - 1 \otimes I_2]. \tag{43}$$

Denote by $\pi_{\varepsilon_{-m+1}}$ the irrep of $gl(m/\infty)$ afforded by $V(\varepsilon_{-m+1})$. The weight spectrum for the vector module $V(\varepsilon_{-m+1})$ consists of all weights ε_i, $i = -m+1, -m+2, \ldots$, each occurring exactly once. Denote by e_{ij}, $i, j = -m+1, -m+2, \ldots$ the generators on this space

$$\pi_{\varepsilon_{-m+1}}(E_{ij}) = e_{ij} \tag{44}$$

with e_{ij} an elementary matrix.

Introduce the characteristic matrix

$$A = \tfrac{1}{2}(\pi_{\varepsilon_{-m+1}} \otimes 1)[\Delta(I_2) - I_2 \otimes 1 - 1 \otimes I_2]. \tag{45}$$

Therefore

$$A_k{}^l = \sum_{i,j=-m+1}^{\infty} (-1)^{(\langle i \rangle + \langle j \rangle)\langle l \rangle} \pi_{\varepsilon_{-m+1}}(E_{ij})_{kl} (-1)^{\langle j \rangle} E_{ji} = (-1)^{\langle k \rangle \langle l \rangle} E_{lk}. \tag{46}$$

The matrix A is the infinite matrix introduced in section 2 (see equation (14)) and the entries of the matrix powers A^q are given recursively by (15). We will see that the characteristic matrix satisfies a polynomial identity acting on the $gl(m/\infty)$ module $V(\Lambda)$, $\Lambda \in D_k^+$. Let π_Λ be the representation afforded by $V(\Lambda)$. From equation (45), acting on $V(\Lambda)$ we may interpret A as an invariant operator on the tensor product module $V(\varepsilon_{-m+1}) \otimes V(\Lambda)$:

$$A \equiv \tfrac{1}{2}(\pi_{\varepsilon_{-m+1}} \otimes \pi_\Lambda)[\Delta(I_2) - I_2 \otimes 1 - 1 \otimes I_2]. \tag{47}$$

Following [11] it is easy to see that the tensor product space admits a filtration of submodules

$$V(\varepsilon_{-m+1}) \otimes V(\Lambda) = V_{k+1} \supseteq V_k \supseteq \ldots V_0 \supseteq \ldots \supseteq V_{-m+1} \supseteq (0) \tag{48}$$

where each factor module $M_i = V_i/V_{i+1}$, if non-zero, is indecomposable and cyclically generated by a highest-weight vector of weight $\Lambda + \varepsilon_i$. We emphasize that M_i is only non-zero when $\Lambda + \varepsilon_i$ is integral dominant. Then it follows that the generalized eigenvalues of A on the tensor product space are given by

$$\begin{aligned} \tfrac{1}{2}[\chi_{\Lambda+\varepsilon_i}(I_2) - \chi_{\varepsilon_{-m+1}}(I_2) - \chi_\Lambda(I_2)] &= \tfrac{1}{2}[(\Lambda + \varepsilon_i, \Lambda + \varepsilon_i + 2\rho) - (\varepsilon_{-m+1}, \varepsilon_{-m+1} + 2\rho) \\ &\quad - (\Lambda, \Lambda + 2\rho)] \\ &= (-1)^{\langle i \rangle}(\Lambda_i + 1 - i) - m \end{aligned} \tag{49}$$

(see theorem 2). Thus we have

Theorem 3. *On each $gl(m/\infty)$ module $V(\Lambda)$, $\Lambda \in D_k^+$ the characteristic matrix satisfies the polynomial identity*

$$\prod_{i=-m+1}^{k+1} (A - \alpha_i) = 0 \tag{50}$$

with $\alpha_i = (-1)^{\langle i \rangle}(\Lambda_i + 1 - i) - m$ the characteristic roots.

Note that the characteristic identities (50) are the $gl(m/\infty)$ counterpart of the polynomial identities encountered for $gl(m/n)$ by Jarvis and Green [12] (more precisely their adjoint identities).

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